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Uncertainty in the Black-Litterman Model - A Practical Note

Adrian Fuhrer
Thorsten Hock

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Adrian Fuhrer[a]

Thorsten Hock[b]

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[a]Quantitative Analyst Asset Management, Pension Fund SBB, Zieglerstrasse 29, CH-3000 Bern 65, adrian.fuhrer@pksbb.ch

[b]Professor of Finance, Consultant Pension Fund SBB, OTH Amberg-Weiden, t.hock@oth-aw.de

Abstract

Die optimale Vermögensallokation von institutionellen Investoren hängt entscheidend von der Qualität der Inputdaten ab, die in den Optimierungsprozess einfließen. Wenn die erwarteten Renditen und die erwartete Kovarianz-Matrix bekannt sind, dann führt die klassische Mean-Varianz-Optimierung nach Markowitz (1952) zu effizienten Portfolios. Falls die Inputfaktoren allerdings nur mit Unsicherheit geschätzt werden können, dann tendiert die Mean-Varianz-Optimierung zu einer Maximierung der Schätzfehler (Michaud, 1989).

Das Black-Litterman-Modell (Black and Litterman (1991, 1992)), ein aus den wissenschaftlichen Methoden der bayesianischen Statistik hergeleiteter Ansatz, ist bei institutionellen Investoren in der praktischen Vermögensallokation weit verbreitet. Es erlaubt die Integration von Rendite-Prognosen und deren Unsicherheit. Beide Größen können mit den Renditen des Marktgleichgewichts kombiniert und konsistent zu modifizierten Erwartungen bezüglich der Renditen und deren Kovarianz-Matrix weiterverarbeitet werden. Diese angepassten Parameter dienen dann als Ausgangspunkt für die Portfolio-Optimierung. Im Black-Litterman-Modell wird die Unsicherheit bezüglich der Gleichgewichtsrenditen ausschließlich mit dem Parameter τ spezifiziert, der ein Skalar darstellt und sehr schwierig zu bestimmen ist. Bei der praktischen Anwendung des Ansatzes führt diese Restriktion zu einem Spezifikationsproblem und zu einem hohen Maß an Einschränkung.

In der vorliegenden Arbeit schlagen wir eine Modifikation des Black-Litterman-Ansatzes vor, die eine flexible Modellierung der Parameterunsicherheit erlaubt. Dies gilt sowohl für die mit den individuellen Prognosen als auch für die mit den Gleichgewichtsrenditen verbundene Unsicherheit. Die vorgeschlagene Anpassung ist ein "Add-on" für das traditionelle Black-Litterman-Modell, die Flexibilität eröffnet und den traditionellen Ansatz als Spezialfall integriert.

Abstract

Deriving an optimal asset allocation for institutional investors hinges crucially on the quality of inputs used in the optimization. If the mean vector $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$ are known with certainty, the classical mean-variance optimization of Markowitz (1952) produces optimal portfolios. If, however, both $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are estimated with uncertainty, mean-variance optimization tends to maximize estimation error, as shown in Michaud (1989).

The *Black-Litterman model* (Black and Litterman (1991, 1992)), a derivation of the Bayesian methods developed in academia, has particular practical appeal for institutional investors. It allows the specification of *views* and an uncertainty about these views, which are combined with equilibrium returns and incorporated consistently to arrive at $\tilde{\boldsymbol{\mu}}$ and $\tilde{\boldsymbol{\Sigma}}$. These new parameters can then be used in the portfolio optimization process. In the Black-Litterman model, however, uncertainty about the equilibrium returns is specified with an overall scalar uncertainty parameter τ, which is difficult to set and introduces rigidity.

We propose a slight modification of the Black-Litterman model to allow the specification of uncertainty in a flexible way not only in individual views, but also in the equilibrium returns of every asset entering the model. Our modification is an “add-on” to the traditional framework, which allows to adjust the uncertainty individually and is still permitting the Black-Litterman approach as a special case.

Contents

1 Introduction

The classical mean-variance optimization of Markowitz (1952) is often considered the standard theoretical model to derive an optimal asset allocation. From a practical standpoint, however, several problems of unconstrained mean-variance optimization have been well documented. The problems predominately stem from the assumption that the mean vector $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$ used for mean-variance optimization are stable and known with certainty. In practice, these problems are often circumvented by imposing additional constraints on individual assets' weights to generate more "intuitive" portfolios, resulting in theoretically inferior diversification. Two competing methods to incorporate parameter uncertainty have emerged from academia (Harvey et al. (2008)). The first one uses a resampling approach, a method developed in Michaud (1998) and Michaud and Michaud (2008). The second one uses Bayesian methods to derive updated distributions of returns (Rachev et al. (2008)). Both methods have been shown to result in portfolios with less concentration, more stability and better out-of-sample performance. They are also not mutually exclusive, as estimates obtained using Bayesian methods can subsequently be used as inputs for the resampling procedure (as done, for instance, in Becker et al. (2015) and Fernandes et al. (2012)).

For many investors however, these methods still lack a convenient way of expressing their subjective views. This has been addressed in seminal work by Black and Litterman (1991, 1992), where they develop the Black-Litterman model, which allows an investor to specify views on some assets. These views are then combined with a market equilibrium using Bayesian methods. It is a widely used model, as it provides a sophisticated statistical framework for practitioners, while allowing for discretion in specifying subjective views. Although the implementation of the Black-Litterman model is conceptually tractable, it hinges on the specification of two parameters of uncertainty; τ, the confidence an investor has in the market equilibrium, and $\boldsymbol{\Omega}$, the confidence he has in his own views. These parameters have been the focus of attention of several subsequent publications, either trying to derive them from data (Scherer (2010), Peterson (2012)) or lending intuition for how to set them (Walters (2010)).

We propose a generalization of the Black-Litterman model with respect to the market equilibrium returns used to anchor the investors' views. More specifically, we replace the confidence parameter τ of the original model with intuitive uncertainty parameters for each asset. These can be elicited from the investor or can be derived from an equilibrium market model directly, while still permitting the classical Black-Litterman model as a special case.

The paper is structured as follows. In Section 2, the literature on parameter uncertainty will be briefly reviewed, with a focus on the Black-Litterman model and proposed extensions. Section 3 will first reproduce the Black-Litterman model, before Section 4 outlines the proposed generalization. Section 5 will illustrate the generalized model and show how the Black-

Litterman model is encompassed as a special case, before Section 6 concludes the paper.

2 Literature Review

While the seminal work by Markowitz (1952) laid the foundations for modern portfolio theory, implementation by practitioners has not been widespread (Michaud (1989) dubbed this the "Markowitz optimization enigma"). Resulting portfolios are often highly-concentrated, very sensitive to the input parameters and maximize estimation error in the inputs (Idzorek (2007)). These related problems predominately stem from the assumption that the mean vector $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$ used for mean-variance optimization are stable and known with certainty. In fact, these input parameters are unknown and can only be estimated with uncertainty, which has to be taken into account in portfolio optimization. Jobson and Korkie (1980, 1981) document this problem. They show that simple equal-weighting actually outperforms mean-variance optimization in the presence of estimation uncertainty in input parameters. Michaud (1989) states that mean variance optimization actually maximize estimation risk. He highlights several additional practical problems that are associated with this fact. Also highlighting practical issues with mean-variance optimization, Best and Grauer (1991) show how sensitive portfolio weights react to the estimated mean returns.

2.1 Dealing with parameter uncertainty

Two main approaches have been developed in the literature to deal with parameter uncertainty. On the one hand, Bayesian methods have been developed to derive Bayesian estimators for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. On the other hand, heuristic methods are used to limit the impact of the uncertainty or to encompass it by "resampling". Both will be reviewed briefly.

2.1.1 Bayesian methods

The Bayesian framework allows to optimally combine two sets of information, usually sample and non-sample data (Rachev et al. (2008)). A prior belief is updated with new data (from the sample or other sources of information) and optimally combined to the posterior distribution. As an intuition, Bayesian methods take into account that a view on one parameter of the model affects all other parameters as well.[1]

For the problem of portfolio selection, Bayesian methods are used to derive updated posterior distributions of returns. A large body of research is available on these methods (cf. Rachev et al. (2008)). Here, a classification of the different approaches with some important literature

[1]For a through discussion of the Bayesian framework, consider Rachev et al. (2008) or Scherer (2010).

is provided. Note that any distribution can either be expressed analytically (i.e. it is assumed to follow a certain parameterized form), or it can be represented non-analytically, through numerical methods and sampling. The classification of available approaches made here follows two dimensions: First, whether the prior distribution is specified analytically or not; and second, whether the posterior distribution can be expressed analytically or not:

- **Parametric prior and posterior distributions** The first, and most simple approach that can be counted to this group of methods is the *uninformative* prior (also called the *diffuse* prior). The uninformative prior does not state any other view than that the parameters are estimated with some uncertainty. Most often, Jeffreys' prior (Jeffreys (1961)) is used to specify this very basic view. The resulting posterior distribution has the functional form of a multivariate normal distribution with the same mean as the sample data, but a scaled covariance matrix. Since the scaling parameter is larger than one, the parameter uncertainty introduced through the uninformative prior leads to overall higher uncertainty in the portfolio optimization.

  As an alternative, a set of methods use *informative* priors or, more specifically, the subgroup of *conjugate* priors. Informative priors allow the specification of prior distributions with well-defined properties. In the case of conjugate priors, the prior distributions are of a functional form that allows the posterior distribution to still be analytically obtainable, that is, the posterior still has a well-defined parametric form (see Frost and Savarino (1986)). A particular application of conjugate priors are *shrinkage* estimators, as developed for instance in Stein (1956), James and Stein (1961), Jorion (1986) or Ledoit and Wolf (2003).

  The Black-Litterman model belongs to this group of Bayesian methods, as the prior views and the equilibrium model are analytically defined and so is the posterior distribution[2]. It was proposed by Black and Litterman (1991, 1992) and further discussed by He and Litterman (1999). According to Rachev et al. (2008), the Black-Litterman model is "the single most prominent application of the Bayesian methodology to portfolio selection". It allows to consistently combine two sets of information: A market-equilibrium and the investors subjective views. This explains its practical appeal, as an investor does not need to rely entirely on either a quantitative model, nor a fully views-based framework, but is able to combine the two in a consistent way. To achieve that, the Black-Litterman model starts from an equilibrium asset pricing model (the CAPM), that is true only with a certain confidence. The equilibrium model is then combined with the investor's subjective views on individual assets or long-short portfolios of assets using Bayesian methods.

[2]Note that while the Black-Litterman model is considered to be a Bayesian method, Avramov and Zhou (2010) point to the fact that it is not entirely Bayesian, as the data generating process is not spelled out and the predictive density is not used.

- **Parametric prior, but non-parametric posterior distributions** This group of methods uses *informative*, but *non-conjugate* priors. As a result, views can be specified much more flexibly but, since the priors are no longer conjugate, the posterior distribution is usually not obtainable in an analytical form, and is thus non-parametric. Applications of these methods can be found, for instance, in Markowitz and Usmen (2005) or Harvey et al. (2008) (see also Rachev et al. (2008)).

- **Non-parametric prior and posterior distributions** In this category, neither the prior nor the posterior distribution is analytically specified. An approach of this kind is provided by Meucci (2008), where it is possible to specify "fully flexible views in fully general non-normal markets". It relies on a methodology called *entropy pooling*, a generalization of Bayesian updating.

2.1.2 Heuristic approaches

Heuristic approaches are used by practitioners to deal with the aforementioned problems of mean-variance optimization. Although they are not based on economic theory, they are commonly used.

Weight Constraints The rationale behind constraints on portfolio weights is the observation by Michaud (1989) that mean-variance optimizers are "estimation-error maximizers": They overweight the securities or asset classes that have large estimated returns, negative correlations or small variances. But these are exactly the securities most likely to contain estimation error. Imposing constraints on exactly those securities or asset classes should contain the problems associated with estimation error. Frost and Savarino (1988) discuss this approach in more detail, and Jagannathan and Ma (2003) show that imposing constraints on portfolio weights actually acts in the same way as shrinkage estimators for the covariance matrix do.

Portfolio resampling This approach, proposed by Michaud (1998) and Michaud and Michaud (2008) has gained a lot of attraction by practitioners. It is intuitively better comprehensible than the statistically more sophisticated Bayesian approaches. An excellent review is provided by Scherer (2002). The resampling approach, as described in Michaud (1998), takes the same inputs as classical mean-variance optimization: A vector $\boldsymbol{\mu}$ of expected returns and a matrix $\boldsymbol{\Sigma}$ of covariances. It also assumes returns to follow a multivariate normal distribution defined by these inputs. To encompass parameter uncertainty, the resampling approach then draws a large number of random samples from this multivariate normal distribution. For each of these random samples, a new $\tilde{\boldsymbol{\mu}}_{\boldsymbol{k}}$ and $\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{k}}$ is obtained ($k = 1, \ldots, N$, where N is the number of new samples). Each of these new parameter-pairs is then used as an input into a classical mean-variance optimization that can also have constraints on portfolio weights. As a result, N vectors of optimal portfolio weights $\tilde{\boldsymbol{\omega}}_{\boldsymbol{k}}$ are obtained. The resampled optimal portfolio is then simply the mean vector over all N weight vectors. Portfolio resampling is thus, in essence, classical mean-variance optimization repeated a large number of times with slightly varying, simulated inputs.

As pointed out by Michaud and Michaud (2008) and Scherer (2002), resampling has various appealing features. First, it produces portfolios that are better diversified and have a lower sensitivity to input parameters (less sudden shifts). Also, since the distribution of portfolio weights is available, estimation error is visualized and can be used, for instance, to implement a rebalancing approach. Scherer (2002) however notes that "there is no economic rationale derived" and points to other problems, especially in the case where no constraints on portfolio weights are present.

2.2 Evaluation

Several publications investigate the performance of the presented approaches to incorporate parameter uncertainty in the portfolio allocation process. Wolf (2006) finds both resampling and shrinkage estimators to outperform classical mean-variance optimization. Markowitz and Usmen (2005), Fernandes and Ornelas (2009), Scherer (2006) and Harvey et al. (2008) compare various Bayesian methods to resampling, with ambiguous results. Fernandes et al. (2012) propose to combine the Black-Litterman model with resampling and actually show that this combined method outperforms in some cases. The largest study in this field is Becker et al. (2015), where seven different Bayesian estimators are compared to their resampled counterparts. They find the resampled versions to perform worse than the not-resampled equivalents.

3 Theory

In a first step, this section will develop the classical Black-Litterman model as published in Black and Litterman (1991, 1992) and He and Litterman (1999)[3]. Then, the deviations proposed to generalize the model are outlined.

[3]As the foundations of the model in Black and Litterman (1991, 1992) and He and Litterman (1999) are somewhat incomplete, there are several papers investigating the model. For instance, Satchell and Scowcroft (2007), Idzorek (2007) and Cheung (2010) provide additional details and mathematical proofs, in addition to several extensions. Consider also Rachev et al. (2008) and Scherer (2010) for a more application-oriented focus.

3.1 The Black-Litterman model

In the Black-Litterman model, the asset returns $\boldsymbol{R}$ (a $(N \times 1)$ vector of random variables, where N is the number of assets) are assumed to come from a multivariate normal distribution:

$$\boldsymbol{R} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \tag{1}$$

where $\boldsymbol{\mu}$ is the $(N \times 1)$ vector of mean returns, and $\boldsymbol{\Sigma}$ is the $(N \times N)$ covariance-matrix. The parameter $\boldsymbol{\mu}$ is expected to contain estimation error, as it is unknown. The Black-Litterman model provides adjusted parameters that incorporate this fact, such that the posterior distribution is

$$\boldsymbol{R} \sim N(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}}) \tag{2}$$

In what follows, the derivation of $\tilde{\boldsymbol{\mu}}$ and $\tilde{\boldsymbol{\Sigma}}$ in the Black-Litterman model is discussed.

Equilibrium Returns In the classical Black-Litterman model, the $(N \times 1)$ vector of equilibrium returns $\boldsymbol{\pi}$ is derived from the CAPM:

$$\boldsymbol{\pi} = \boldsymbol{\beta}(R_M - R_f), \tag{3}$$

where $\boldsymbol{\beta}$ is the $(N \times 1)$ vector of market betas of the assets. Although the market is expected to be in equilibrium on average, at any given point in time, it could be in disequilibrium. Therefore,

$$\boldsymbol{\mu} = \boldsymbol{\pi} + \boldsymbol{\epsilon} \tag{4}$$

$$\text{with} \qquad \boldsymbol{\epsilon} \sim N(\mathbf{0}, \boldsymbol{\Psi})$$

$$\text{and} \qquad \boldsymbol{\Psi} = \tau\boldsymbol{\Sigma}, \tag{5}$$

where $\boldsymbol{\epsilon}$ is a $(N \times 1)$ vector of random shocks that push the market off its long-run equilibrium and $\boldsymbol{\Psi}$ is the $(N \times N)$ covariance matrix of these shocks.

Combining these yields the *prior distribution* of $\boldsymbol{\mu}$:

$$\boldsymbol{\mu} \sim N(\boldsymbol{\pi}, \tau\boldsymbol{\Sigma}) \tag{6}$$

The prior covariance matrix is simply the scaled covariance matrix of the sampling distribution. The scaling parameter τ represents the uncertainty in the accuracy with which $\boldsymbol{\pi}$ is estimated. It is an important hyperparameter and the main focus of the generalisation presented later in the paper.

Investor Views One of the main advantages of the Black-Litterman model is the possibility for the investor to specify subjective views on the absolute or relative performance of assets. These views are specified using the views matrix $\boldsymbol{P}$, an $(K\times N)$-Matrix of K views on N assets; a vector $\boldsymbol{q}$ of expected returns of the K views; and a $(K\times K)$-Covariance Matrix $\boldsymbol{\Omega}$ of these views. $\boldsymbol{P}$ and $\boldsymbol{q}$ are of the following form:

$$\boldsymbol{P} = \begin{bmatrix} p_{1,1} & \cdots & p_{1,N} \\ \vdots & \ddots & \vdots \\ p_{K,1} & \cdots & p_{K,N} \end{bmatrix} \qquad \boldsymbol{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_K \end{bmatrix}$$

Every line of $\boldsymbol{P}$ specifies a long-short (or long only) portfolio of the N assets, with every $p_{k,n}$ specifying the weight of the nth asset in the kth view. Every element of $\boldsymbol{q}$ then specifies the expected return of the respective portfolio. The returns of the views are also assumed to be uncertain, such that:

$$\boldsymbol{P}\boldsymbol{\mu} = \boldsymbol{q} + \boldsymbol{\varepsilon} \tag{7}$$

$$\text{where} \qquad \boldsymbol{\varepsilon} \sim N(\boldsymbol{0}, \boldsymbol{\Omega}),$$

with $\boldsymbol{\varepsilon}$ a $(K\times 1)$ vector of random shocks.

For the covariance matrix of the shocks, a simplifying assumption is usually made: The views are assumed to be uncorrelated, i.e. the covariance matrix $\boldsymbol{\Omega}$ is assumed to be diagonal (with zeros on all off-diagonal elements):

$$\boldsymbol{\Omega} = \begin{bmatrix} \omega_{1,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_{k,k} \end{bmatrix} \tag{8}$$

The elements $\omega_{k,k}$ encompass the uncertainty about the views. They should be inversely proportional to the strength of the investor's confidence in the kth view. Different ways to determine $\omega_{k,k}$ will be discussed in the Section 3.2.

With $\boldsymbol{P}$, $\boldsymbol{q}$ and $\boldsymbol{\Omega}$ thus defined, the *prior distribution* of the view's expected returns is assumed to be a multivariate normal distribution of the form:

$$\boldsymbol{P}\boldsymbol{\mu} \sim N(\boldsymbol{q}, \boldsymbol{\Omega}) \tag{9}$$

Combining the prior distributions Using Bayes' theorem, the two sources of information can be combined consistently. The posterior distribution of expected returns $\boldsymbol{\mu}$ is[4]

$$\boldsymbol{\mu} \sim N(\boldsymbol{m}, \boldsymbol{V}), \tag{10}$$

where

$$\boldsymbol{m} = \boldsymbol{V}\left(\boldsymbol{\Psi}^{-1}\boldsymbol{\pi} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{q}\right) \tag{11}$$

and

$$\boldsymbol{V} = \left(\boldsymbol{\Psi}^{-1} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{P}\right)^{-1} \tag{12}$$

The posterior distribution of the assets' returns are then (as outlined above):

$$\boldsymbol{R} \sim N(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}}) \tag{13}$$

$$\text{with} \quad \tilde{\boldsymbol{\mu}} = \boldsymbol{m} \qquad \text{and} \qquad \tilde{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma} + \boldsymbol{V}$$

Note that the new covariance matrix $\tilde{\boldsymbol{\Sigma}}$ is simply the sample covariance matrix $\boldsymbol{\Sigma}$ increased by the uncertainty surrounding the estimate of $\tilde{\boldsymbol{\mu}}$, captured by $\boldsymbol{V}$.

Next, the specification of the uncertainty of the investor towards the model and the views is discussed.

3.2 Setting Hyperparameters

In the Black-Litterman model, two hyperparameters are used to specify the investors confidence in (a) the equilibrium returns and (b) his own views. They are captured by $\boldsymbol{\Psi}$ and $\boldsymbol{\Omega}$, respectively. More generally, these parameters express the uncertainty about the expected equilibrium returns $\boldsymbol{\pi}$ and the expected returns of the view-portfolios $\boldsymbol{q}$ that enter the model. As pointed out by Idzorek (2007), setting these parameters are "the most abstract and difficult to specify parameters of the model". As a consequence, several different methods to set these parameters have emerged from the literature. A large part of this literature is focused on setting $\boldsymbol{\Omega}$, the uncertainty about the expected returns of the views. While the present paper focuses on $\boldsymbol{\Psi}$, it is still useful to review these ideas, as variants thereof will be used later to lend intuition about the proposed way of setting $\boldsymbol{\Psi}$.

Setting Hyperparameter $\boldsymbol{\Psi}$ In Black and Litterman (1992), the original authors reduce the problem of setting $\boldsymbol{\Psi}$ to setting a single scalar parameter τ, as they propose that $\boldsymbol{\Psi}$ is proportional to the covariance matrix $\boldsymbol{\Sigma}$ (see Equation (5)). The authors recommend to use a

[4]For a mathematical proof, consider for instance Satchell and Scowcroft (2007).

τ that is smaller than one and close to zero, as the mean of expected returns can much more accurately be determined than the expected returns themselves. Idzorek (2007) interprets τ as the inverse of the relative weight given to the equilibrium weights, or alternatively, inverse to the degree of belief in the equilibrium model. He also reports that practitioners recommend using a value of τ between 0.01 and 0.05. On the other hand, Satchell and Scowcroft (2007) propose to use a value of 1 for τ. Finally, Rachev et al. (2008) and Meucci (2010) propose to use the standard error of the estimate of the implied equilibrium return directly, which is approximately 1 divided by the number of observations. They do, however, also state that "no guideline exists for the selection of their values".
Although some suggestions of how to set τ are available, there is still a lot of subjectivity. In our view, it would be beneficial to either make this subjectivity more explicit or to derive empirical methods to determine τ.[5] In order to do this, the ideas developed to specify $\mathbf{\Omega}$, the uncertainty about the expected returns of the views will be reviewed next, as they might serve as the basis to solve the issue.

Setting Hyperparameter $\mathbf{\Omega}$ For setting $\mathbf{\Omega}$ too, Black and Litterman (1991, 1992) make a simplifying assumption: As views are assumed to be independent of each other, $\mathbf{\Omega}$ reduces to a diagonal matrix (as shown in Equation (8)). They propose to express the uncertainty or, conversely, the confidence in a view as the number of observations drawn from the distribution of future returns. Alternatively, if the view is assumed to directly specify a probability distribution, a variance or volatility of the view can be specified.
Several alternative procedures have been proposed to estimate $\mathbf{\Omega}$:

- **He and Litterman (1999)** In order to determine each element $\omega_{k,k}$ of $\mathbf{\Omega}$, He and Litterman (1999) propose to use the same τ as used in the estimation of $\mathbf{\Psi}$ (see above). The variance of the kth view portfolio is simply $\boldsymbol{p}_k \mathbf{\Sigma} \boldsymbol{p}_k'$.[6] It is then scaled with the same constant τ, thus resulting in:
$$\omega_{k,k} = \tau \boldsymbol{p}_k \mathbf{\Sigma} \boldsymbol{p}_k' \tag{14}$$
This reduces the complexity of the model, as only the parameter τ has to be specified by the investor, but also reduces flexibility, as the confidence in the equilibrium model is also forced on the confidence in the investor's views.

- **Satchell and Scowcroft (2007)** Using Bayesian methods, Satchell and Scowcroft (2007) allow for a prior belief on the covariance matrix as well.

- **Idzorek (2007)** In order to determine $\mathbf{\Omega}$, Idzorek (2007) uses an iterative process that

[5] In the Appendix A.1, we provide an interesting alternative way of interpreting τ.
[6] $\boldsymbol{p}_k$ is the kth row of the views matrix $\boldsymbol{P}$

uses certainty-equivalent weights and chooses the portfolio weights such that they represent a confidence specified by the investor. This procedure implies an unconstrained portfolio optimization.

- **Peterson (2012)** Limiting the flexibility of the model, Peterson (2012) leaves no degree of freedom in the estimation of $\mathbf{\Omega}$ and uses the approach of He and Litterman (1999), but without the input parameter τ:

$$\omega_{k,k} = \boldsymbol{p}_k \mathbf{\Sigma} \boldsymbol{p}_k' \tag{15}$$

While the simplicity of this approach is appealing, it seems reasonable to allow the investor to specify some form of confidence in her views.

- **Judgmental Approach (Rachev et al. (2008); Scherer (2010))** If Peterson (2012), relying entirely on the data and leaving no room for the investor to specify a confidence in views, is on one end of the spectrum, then the judgemental approach proposed in Rachev et al. (2008) and Scherer (2010) is on the other end. Here, implied variances of views are derived entirely from the investors judgement. As the Black-Litterman model assumes the returns of the views-portfolios to be independently normally distributed (i.e. $\boldsymbol{P\mu} \sim N(\boldsymbol{q}, \mathbf{\Omega})$, with $\mathbf{\Omega}$ a diagonal matrix), the investor can be asked to characterize the normal distribution of each view. To do this, in addition to the expected return of the view, she is asked to specify an interval in which the returns are expected to be, as well as a confidence that they will actually be in this interval.

If $(1-\alpha_k)$ denotes the investors confidence in the kth view (for a 95% confidence, $\alpha = 5\%$), q_k specifies the expected return of the view (the kth element in $\boldsymbol{q}$) and l_k is the investor-specified lower limit, then

$$\omega_{k,k} = \left(\frac{l_k - q_k}{Z_{\alpha_k}} \right)^2, \tag{16}$$

where Z_{α_k} is the (α_k)-quantile of the standard normal distribution. Note that the specified interval needs to be symmetric, such that the upper limit is defined as $u_k = q_k + (q_k - l_k)$.

- **Scherer (2010)** An interesting alternative is proposed by Scherer (2010), which can be used if views are derived from a quantitative forecasting model: The diagonal elements of $\omega_{k,k}$ are then simply the unexplained variances of the respective forecasting model:

$$\mathbf{\Omega} = \begin{bmatrix} \sigma_1^2\,(1-R_1^2) & 0 & \cdots & 0 \\ 0 & \sigma_2^2\,(1-R_2^2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_k^2\,(1-R_k^2) \end{bmatrix} \tag{17}$$

Reviewing the literature on how to set the hyperparameters of the Black-Litterman model shows that considerably more work has been published on how to set the uncertainty in the expected returns of the views than in the expected equilibrium returns. The aim of the next part is to show how the methods developed to more accurately specify $\mathbf{\Omega}$ can also be used to determine $\mathbf{\Psi}$ is a more intuitive and flexible way.

4 Deviation from the Black-Litterman model

As discussed in Part 3.2, setting the hyperparameters of the Black-Litterman model poses a challenge for practical applications. While several approaches are available to flexibly specify the uncertainty about the expected returns of the views $\mathbf{\Omega}$, uncertainty about the equilibrium returns $\mathbf{\Psi}$ is specified through a single parameter τ that forces proportionality to the covariance matrix $\mathbf{\Sigma}$.[7] We propose a simple deviation of the Black-Litterman model to allow a more flexible specification of $\mathbf{\Psi}$, and make two propositions of how to implement this practically. These are adaptions of two ideas proposed to determine $\mathbf{\Omega}$, applied to the specification of $\mathbf{\Psi}$.

4.1 Introducing Flexibility

The classical Black-Litterman model assumes that the uncertainty about equilibrium returns is proportional to the uncertainty about the returns themselves (Black and Litterman (1992)), as formalized in Equation (5).

We would like to allow for more flexibility, still encompassing the Black-Litterman specification as a special case. In order to do this, first decompose the covariance matrix of the returns $\mathbf{\Sigma}$ into a $(N \times 1)$ volatility vector $\boldsymbol{\sigma}$ and a $(N \times N)$ correlation matrix $\mathbf{\Phi}$:

$$\mathbf{\Sigma} = \text{diag}(\boldsymbol{\sigma})\mathbf{\Phi}\text{diag}(\boldsymbol{\sigma}) \qquad \text{or} \qquad \mathbf{\Phi} = \text{diag}(\boldsymbol{\sigma})^{-1}\mathbf{\Sigma}\text{diag}(\boldsymbol{\sigma})^{-1} \tag{18}$$

This allows us to retain the information about the co-movement of equilibrium returns, but leaves flexibility in their uncertainty. Define the vector of standard errors of estimated equilibrium returns chosen by the investor as $\hat{\boldsymbol{\sigma}}$, then

$$\mathbf{\Psi} = \text{diag}(\hat{\boldsymbol{\sigma}})\mathbf{\Phi}\text{diag}(\hat{\boldsymbol{\sigma}}) \tag{19}$$

To see how this is a generalization of the Black-Litterman approach, suppose the investor sets

[7] While the Black-Litterman model proposes to use the CAPM as the equilibrium model, in practice a variety of equilibrium models can be employed. Common choices are: Historical analysis, Econometric models, Valuation based analysis, amd Market based measures. While these models yield different results, they are not the subject of this paper. They can, however, be used as a basis and integrated consistently into the flexible framework developed here, as outlined in Section 4.2.

$\hat{\boldsymbol{\sigma}} = \sqrt{\tau}\boldsymbol{\sigma}$ (i.e. he sets the standard errors of estimated equilibrium returns to be proportional to the empirical volatilities of returns). Then, we recover

$$\boldsymbol{\Psi} = \text{diag}(\hat{\boldsymbol{\sigma}})\boldsymbol{\Phi}\text{diag}(\hat{\boldsymbol{\sigma}}) = \text{diag}(\sqrt{\tau}\boldsymbol{\sigma})\boldsymbol{\Phi}\text{diag}(\sqrt{\tau}\boldsymbol{\sigma}) = \tau\boldsymbol{\Sigma} \tag{20}$$

but do not restrict the investor to this case.

With this decomposition, the investor has the ability to specify the uncertainty about equilibrium returns for each asset individually. How this can benefit practitioners will be shown with two applications.

4.2 Empirical Specification through Model-Approach

In the Black-Litterman model, the equilibrium returns are derived from the CAPM. They are, however, not usually obtained by a regression analysis, but are backed-out through the knowledge of the $(N\times1)$ weights of the market portfolio $\boldsymbol{\omega}_{eq}$:

$$\boldsymbol{\pi} = \delta\boldsymbol{\Sigma}\boldsymbol{\omega}_{eq} \tag{21}$$

where $\delta = (R_M - R_f)/\sigma_M^2$ (consider Rachev et al. (2008) for more details). This requires that the weights of the market portfolio $\boldsymbol{\omega}_{eq}$ are known for each asset, which is typically not the case for an asset allocation process in the multi asset class framework, especially when alternative asset classes are considered as well.

In that case, it is usually a requirement to revert to a regression analysis for the equilibrium model. It is then straightforward to use the approach proposed by Scherer (2010) for the specification of uncertainty about the expected returns of the views discussed in Part 3.2 also here.

If a suitable reference portfolio is specified, in the CAPM framework, the excess returns of each asset class are regressed on the excess returns of the reference portfolio. The uncertainty of the estimated equilibrium return is then, in the sense proposed by Scherer (2010), the unexplained variance of this regression:

$$\sigma_{\pi,k} = \sqrt{\sigma_k^2\,(1 - R_k^2)} \tag{22}$$

The simplicity of this approach opens up a lot of possibilities. It allows the use of different equilibrium models for each asset class, for instance more sophisticated multi-factor models.

4.3 Intuition through Judgemental Approach

In order to increase the intuition in specifying the uncertainty in the expected equilibrium returns, we follow and generalise the *Judgemental Approach* proposed in Rachev et al. (2008) and Scherer (2010) to specify uncertainty in the expected equilibrium returns. What follows is our proposition to model this uncertainty flexibly and consistently, and to integrate it into the Black-Litterman model.

Symmetric Confidence Intervals First, consider the case of symmetric confidence intervals. For each asset, the parameters of interest are π and σ_π. The relationship of the confidence level α, the upper and lower limits u_π and l_π and the parameters of interest is depicted in Figure 1.

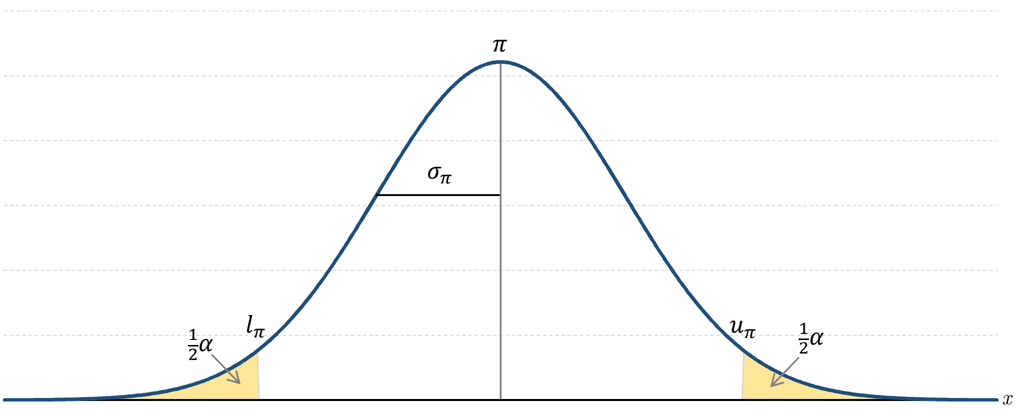

Figure 1: Symmetric confidence intervals of a normal distribution. π is the expected return, σ_π the standard deviation of the expected return, α the confidence level (for a 95% confidence level, $\alpha = 5\%$) and l_π and u_π are the lower and upper limit, respectively

There are two fundamental relations between the parameters:

$$\pi = \frac{u_\pi + l_\pi}{2} \tag{23}$$

and

$$\sigma_\pi = \frac{l_\pi - \pi}{Z_{\alpha/2}}, \tag{24}$$

where $Z_{\alpha/2}$ is the $(\alpha/2)$-quantile of the standard normal distribution.

With these, the investor can be asked to specify various combinations of parameters to arrive at π and σ_π. The most convenient ones are tabulated in Table 1. In the first and second line of Table 1 the investor specifies his expectations about the mean return, and either a lower or an upper limit that the mean return is expected to lie in with a confidence level $(1 - \alpha)$.

User-specified	π	σ_π
π, l_π, α	π	$\frac{l_\pi - \pi}{Z_{\alpha/2}}$
π, u_π, α	π	$\frac{\pi - u_\pi}{Z_{\alpha/2}}$
l_π, u_π, α	$\frac{u_\pi + l_\pi}{2}$	$\frac{l_\pi - u_\pi}{Z_{\alpha/2}}$

Table 1: Different parameter-combinations elicited form the investor to compute π and σ_π.

The last line of Table 1 is probably the most intuitive: The investor simply states the interval the expected equilibrium return lies in, with a confidence in that interval.

Asymmetric Confidence Intervals Since the expected mean returns are required to follow a normal distribution, introducing asymmetry in the estimate is not possible in this framework. It is, however, possible for the investor to specify an asymmetric confidence interval. The situation is depicted in Figure 2 and is very similar to the situation in Figure 1 with symmetric confidence intervals. But here, the probability mass in the tails (α_l and α_u) can be specified

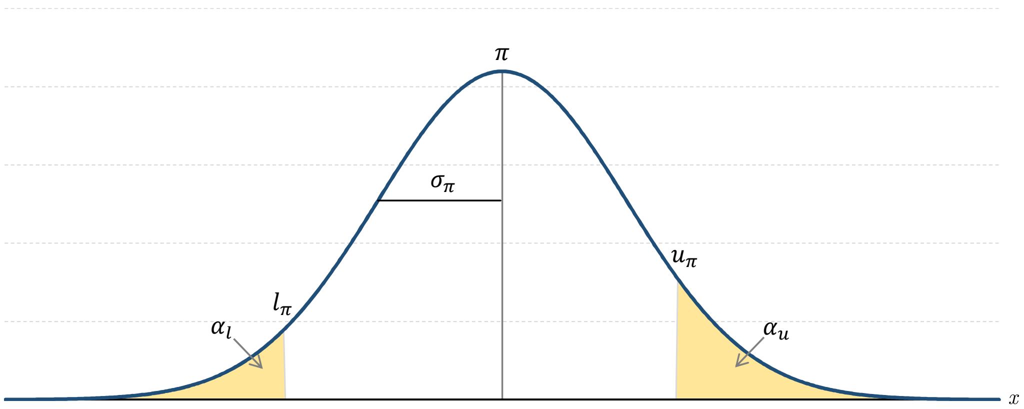

Figure 2: Asymmetric confidence intervals of a normal distribution. Notation is as in Figure 1, with the exception that α is allowed to be asymmetric ($\alpha = \alpha_l + \alpha_u$, with $\alpha_l \neq \alpha_u$). The investor can specify the probability mass in the tails independently.

independently. Then,

$$\pi = \frac{u_\pi Z_{\alpha_l} + l_\pi Z_{1-\alpha_u}}{Z_{\alpha_l} - Z_{1-\alpha_u}} \tag{25}$$

and

$$\sigma_\pi = \frac{l_\pi - \pi}{Z_{\alpha_l}}. \tag{26}$$

Correspondence to τ As the parameter τ used in the classical Black-Litterman model is simply the parameter of proportionality between $\mathbf{\Psi}$ and $\mathbf{\Sigma}$, it can be computed from the specified $\boldsymbol{\sigma}_\pi$. For each asset k, given the empirical variance of the returns σ_k^2 and the specified variance of the estimate of the equilibrium return, $\sigma_{\pi,k}^2$, τ is simply:

$$\tau_k = \frac{\sigma_{\pi,k}^2}{\sigma_k^2} \tag{27}$$

Note that we index τ over k as well, as with the added flexibility, τ can be different for every asset.
On the other hand, this correspondence also allows the computation of implicit confidence intervals from the assumptions of the Black-Litterman model. Given $\boldsymbol{\pi}$, $\mathbf{\Sigma}$ and τ from the model, and assuming a confidence level of α, the corresponding symmetric confidence intervals can readily be obtained as:

$$[\boldsymbol{l}_\pi, \boldsymbol{u}_\pi] = \boldsymbol{\pi} \mp \tau \cdot \text{diag}(\mathbf{\Sigma}) \cdot Z_{\alpha/2} \tag{28}$$

For asymmetric confidence intervals, with α_l and α_u specified by the investor, the interval is defined as:

$$\boldsymbol{l}_\pi = \boldsymbol{\pi} - \tau \cdot \text{diag}(\mathbf{\Sigma}) \cdot Z_{\alpha_l} \tag{29}$$

and

$$\boldsymbol{u}_\pi = \boldsymbol{\pi} + \tau \cdot \text{diag}(\mathbf{\Sigma}) \cdot Z_{1-\alpha_u} \tag{30}$$

We propose to use these intervals as a starting point for the investor. From there, she can adjust the intervals and confidence according to her knowledge about the models generating the estimated equilibrium returns.

5 Illustration

To illustrate how the generalised model can be used to derive the updated input parameters for portfolio optimization and the resulting asset allocation, a simple four-asset example will be introduced in this section. The section is structured as follows: First, the data and expected equilibrium returns are presented. Then, the classical Black-Litterman model with a constant τ but without views is implemented. The resulting allocation serves as the basis of comparison. We then introduce our flexible model, by allowing the investor to specify uncertainty in the equilibrium model other than through the constant τ. Next, we introduce a set of subjective views and recompute both the Black-Litterman model and our flexible model under these views. This allows to identify and discuss the effects of our results.

5.1 A Four-Asset Example

Consider a simple example with four asset classes to choose from: Global equities (GE), global government bonds (GGB), emerging market bonds (EMB) and real estate funds (REF)[8]. Assume that they are characterised by the following historical mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ of risk premia:

	$\boldsymbol{\mu}$
GE	6.43%
GGB	3.26%
EMB	4.76%
REF	9.06%

$\boldsymbol{\Sigma}$	GE	GGB	EMB	REF
GE	1.78%	-0.16%	1.04%	1.31%
GGB	-0.16%	0.23%	-0.13%	0.00%
EMB	1.04%	-0.13%	1.40%	0.89%
REF	1.31%	0.00%	0.89%	3.24%

As suggested in Section 4.1, $\boldsymbol{\Sigma}$ can be decomposed into historical volatilities $\boldsymbol{\sigma}$ and the correlation matrix $\boldsymbol{\Phi}$:

	$\boldsymbol{\sigma}$
GE	13.35%
GGB	4.74%
EMB	11.85%
REF	18.00%

$\boldsymbol{\Phi}$	GE	GGB	EMB	REF
GE	1.00	-0.25	0.66	0.54
GGB	-0.25	1.00	-0.23	0.00
EMB	0.66	-0.23	1.00	0.42
REF	0.54	0.00	0.42	1.00

Suppose that the following equilibrium returns are derived from an equilibrium model (for instance, the CAPM):

	$\boldsymbol{\pi}$
GE	3.50%
GGB	0.60%
EMB	2.50%
REF	3.00%

5.2 Asset Allocation without Views

In a first step, we illustrate how the added flexibility in the proposed model influences the asset allocation when the investor has no subjective views about absolute or relative performances of assets. As a basis of comparison, we first derive the asset allocation using the Black-Litterman model with the constant τ set to 0.05 in Section 5.2.1. Implied confidence intervals are derived and presented. Then, in Section 5.2.2, we will slightly adjust these confidence intervals and, using the flexible model, will derive a new asset allocation. A comparison of the two allocations allows the identification of the effects of the flexible model specification.

[8]Consider Appendix A.2 for details on the time series used.

5.2.1 Using the Black-Litterman Model

To use the Black-Litterman model without investor's subjective views, only the hyperparameter τ has to be set. We choose to set $\tau = 0.05$, which is in line with suggestions summarized in Idzorek (2007). According to Equation (5), the uncertainty about the expected equilibrium returns $\boldsymbol{\Psi}$ is simply the scaled covariance matrix $\boldsymbol{\Psi} = \tau\boldsymbol{\Sigma}$:

$\boldsymbol{\Psi}$	GE	GGB	EMB	REF
GE	0.089%	-0.008%	0.052%	0.065%
GGB	-0.008%	0.011%	-0.007%	0.000%
EMB	0.052%	-0.007%	0.070%	0.045%
REF	0.065%	0.000%	0.045%	0.162%

For illustrative purposes, and following Section 4.1, we decompose $\boldsymbol{\Psi}$ into the volatility vector of the expected equilibrium returns $\boldsymbol{\sigma_\pi}$ and the correlation matrix $\boldsymbol{\Phi}$:

	$\boldsymbol{\sigma_\pi}$
GE	2.99%
GGB	1.06%
EMB	2.65%
REF	4.03%

$\boldsymbol{\Phi}$	GE	GGB	EMB	REF
GE	1.00	-0.25	0.66	0.54
GGB	-0.25	1.00	-0.23	0.00
EMB	0.66	-0.23	1.00	0.42
REF	0.54	0.00	0.42	1.00

As expected, the correlation matrix $\boldsymbol{\Phi}$ is equivalent to the correlation matrix of the data presented in Section 5.1 and the volatility vector of the expected equilibrium returns is simply the scaled volatility vector of the returns $\boldsymbol{\sigma_\pi} = \sqrt{\tau}\boldsymbol{\sigma}$. It is also possible to obtain the implied symmetric confidence intervals of this specification for an assumed confidence level of $(1-\alpha) = 80\%$ (i.e. 10% of probability mass in each symmetric tail), derived using Equation (28):

	$\boldsymbol{\pi}$	l_k^π	u_k^π	$\alpha_{l,k}$	$\alpha_{u,k}$	τ
GE	3.50%	-0.33%	7.33%	10%	10%	0.050
GGB	0.60%	-0.76%	1.96%	10%	10%	0.050
EMB	2.50%	-0.90%	5.90%	10%	10%	0.050
REF	3.50%	-1.66%	8.16%	10%	10%	0.050

All the inputs required to compute the updated parameters $\tilde{\boldsymbol{\mu}}$ and $\tilde{\boldsymbol{\Sigma}}$ from the Black-Litterman model are now defined. Using Equations (11) - (13), the following updated parameters are obtained:

	$\tilde{\boldsymbol{\mu}}$
GE	3.50%
GGB	0.60%
EMB	2.50%
REF	3.50%

$\tilde{\boldsymbol{\Sigma}}$	GE	GGB	EMB	REF
GE	1.87%	-0.16%	1.09%	1.37%
GGB	-0.16%	0.24%	-0.14%	0.00%
EMB	1.09%	-0.14%	1.47%	0.93%
REF	1.37%	0.00%	0.93%	3.40%

	$\tilde{\boldsymbol{\sigma}}$
GE	13.68%
GGB	4.86%
EMB	12.14%
REF	18.45%

These parameters can be used as the inputs of a classical mean-variance optimization problem as proposed by Markowitz (1952). The resulting portfolio weights $\boldsymbol{\omega}_{BL}$ are reported below. The weights are constraint to add to one, the portfolio volatility σ^p is constraint to 8.00% and the expected equilibrium return is maximised.

	$\boldsymbol{\omega}_{BL}$
GE	46.61%
GGB	35.18%
EMB	8.85%
REF	9.35%

5.2.2 Using the Flexible Model

In the previous section, by setting the hyperparameter τ, the investor implicitly specified the same level of confidence in each of the estimated equilibrium returns. This becomes obvious when investigating the implied confidence intervals already derived in Section 5.2.1 above, where τ is constant for all assets.

Suppose, however, that the investor derives these estimates not from a single equilibrium model, but from specific models for each asset, or that experience suggests that equilibrium returns for some assets are more reliably estimated than for others. For instance, the investor might have a smaller confidence in the estimate of the equilibrium return of global equities. We represent this by leaving the upper and lower limit unchanged, but change the mass in the tails from 10% to 20% for each. Our confidence in the estimated equilibrium returns thus decreases from 80% to 60%. The second modification we propose is that the investor has a very specific equilibrium model for emerging market bonds, which suggests a higher confidence in the form of limits closer to the mean and a slightly asymmetric confidence interval[9]. Both modifications are added in the table below:

[9]To insure that the expected equilibrium return of emerging market bonds is unchanged at 2.50%, the probability mass in the lower tail has the somewhat odd value of 11%.

	$\boldsymbol{\pi}$	l_k^{π}	u_k^{π}	$\alpha_{l,k}$	$\alpha_{u,k}$	τ
GE	3.50%	-0.33%	7.33%	20%	20%	0.116
GGB	0.60%	-0.76%	1.96%	10%	10%	0.050
EMB	2.50%	1.00%	4.50%	11%	5%	0.011
REF	3.50%	-1.66%	8.16%	10%	10%	0.050

The parameter τ is no longer constant in this case. It is higher for global equities, representing the higher uncertainty in the equilibrium estimate, and lower for emerging market bonds, as the investor is able to estimate the equilibrium returns more accurately. On average, τ is still very close to 0.05, guaranteeing that results are not driven by additional risk of the equilibrium model.

The new assumptions about the expected equilibrium returns result in the following parameters $\boldsymbol{\sigma_\pi}$ and $\boldsymbol{\Psi}$:

	$\boldsymbol{\sigma_\pi}$
GE	4.55%
GGB	1.06%
EMB	1.22%
REF	4.03%

$\boldsymbol{\Psi}$	GE	GGB	EMB	REF
GE	0.207%	-0.012%	0.036%	0.100%
GGB	-0.012%	0.011%	-0.003%	0.000%
EMB	0.036%	-0.003%	0.015%	0.020%
REF	0.100%	0.000%	0.020%	0.162%

Using Equations (11) - (13) also here, the updated parameters are obtained as follows:

	$\tilde{\boldsymbol{\mu}}$
GE	3.50%
GGB	0.60%
EMB	2.50%
REF	3.50%

$\tilde{\boldsymbol{\Sigma}}$	GE	GGB	EMB	REF
GE	1.99%	-0.17%	1.08%	1.41%
GGB	-0.17%	0.24%	-0.13%	0.00%
EMB	1.08%	-0.13%	1.42%	0.91%
REF	1.41%	0.00%	0.91%	3.40%

	$\tilde{\boldsymbol{\sigma}}$
GE	14.10%
GGB	4.86%
EMB	11.91%
REF	18.45%

With the exact same optimization routine used above ($\sigma^p = 8.00\%$), the new weights are obtained and compared to the Black-Litterman solution below:

	$\boldsymbol{\omega}_{BL}$	$\boldsymbol{\omega}_{FL}$
GE	46.61%	41.34%
GGB	35.18%	34.50%
EMB	8.85%	13.81%
REF	9.35%	10.35%

As would be expected, the added uncertainty about the equilibrium returns of global equities leads to a smaller proportion of wealth invested in this asset class. Conversely, a larger proportion of wealth is invested in emerging market bonds, as the expected equilibrium return is specified with higher confidence. Differences in the weights of the other asset classes stem from the propagation of the effects through the correlation matrix with the Bayesian methodolgoy inherent to the Black-Litterman approach.

5.3 Investor's Subjective Views

In a second comparison, investor's subjective market views are introduced. They are defined as in the original Black-Litterman model. It is assumed that the investor holds three distinct views about the four assets:

1. Global equities (GE) will have a performance of 4% for the foreseeable future.
2. Global government bonds (GGB), on the other hand, have an expected return of only 0.5%.
3. Real estate funds (REF) will outperform global government bonds (GGB) and emerging market bonds (EMB) by 1 percentage point.

From this, both $\boldsymbol{P}$ and $\boldsymbol{q}$ can be obtained:

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \qquad \boldsymbol{q} = \begin{bmatrix} 4.0\% \\ 0.5\% \\ 1.0\% \end{bmatrix}$$

In order to specify the uncertainty about these views, the method proposed by He and Litterman (1999) is used. Accordingly, the diagonal elements of $\boldsymbol{\Omega}$ can be determined as $\omega_{k,k} = \tau \boldsymbol{p}_k \boldsymbol{\Sigma} \boldsymbol{p}_k'$ and the off-diagonal elements are assumed to be zero. We still assume a constant $\tau = 0.05$ for the views, yielding the following views-covariance matrix (using Equation (14)):

$\boldsymbol{\Omega}$	View 1	View 2	View 3
View 1	0.09%	0	0
View 2	0	0.01%	0
View 3	0	0	0.13%

It is important to note that these views are used identically in both the classical Black-Litterman model illustration, as well as in the illustration of the more general model proposed in this paper. This allows to isolate the impact of our modification on the asset allocation when views are present.

5.4 Asset Allocation with Views

The views just introduced will now be implemented in both the classical Black-Litterman model as well as in the more flexible model. To allow a comparison, the setup in both models is kept equivalent to the analysis in Section 5.2, and the same views, as just outlined, are introduced to both models.

5.4.1 Using the Black-Litterman Model

As before, we use Equations (11) - (13) to compute the updated parameters from the Black-Litterman model, this time taking into account the investor's subjective views:

	$\tilde{\boldsymbol{\mu}}$
GE	3.67%
GGB	0.54%
EMB	2.66%
REF	3.16%

$\tilde{\boldsymbol{\Sigma}}$	GE	GGB	EMB	REF
GE	1.82%	-0.16%	1.07%	1.33%
GGB	-0.16%	0.23%	-0.13%	0.00%
EMB	1.07%	-0.13%	1.46%	0.92%
REF	1.33%	0.00%	0.92%	3.32%

	$\tilde{\boldsymbol{\sigma}}$
GE	13.51%
GGB	4.80%
EMB	12.07%
REF	18.22%

The parameter $\tilde{\boldsymbol{\mu}}$ is pulled into the direction of the views. Especially View 3 has a non-trivial influence on the parameter: As it assumes the relative return of global government bonds and emerging market bonds to real estate funds to be smaller than it actually is, the expected return of real estate funds is reduced, while that of global government bonds and emerging market bonds is increased. The effect on global government bonds is however not visible, as there is an interaction with the absolute View 2, targeted directly on this asset class. The resulting volatilities are generally smaller, as the added information reduces the uncertainty in the model.

The resulting asset allocation is reported below and compared to the Black-Litterman model without views:

	$\boldsymbol{\omega}_{BL}$	$\boldsymbol{\omega}_{BL}^{V}$
GE	46.61%	51.70%
GGB	35.18%	33.96%
EMB	8.85%	12.08%
REF	9.35%	2.25%

As can be seen, the allocation weights are pulled into the direction indicated by the views.

5.4.2 Using the Flexible Model

The same experiment is repeated for the more flexible model. All specifications are equivalent to the specifications in Section 5.2.2, but the views as outlined in Section 5.3 are introduced. The resulting parameters are the following:

	$\tilde{\boldsymbol{\mu}}$
GE	3.75%
GGB	0.54%
EMB	2.55%
REF	3.13%

$\tilde{\boldsymbol{\Sigma}}$	GE	GGB	EMB	REF
GE	1.84%	-0.16%	1.05%	1.32%
GGB	-0.16%	0.23%	-0.13%	0.00%
EMB	1.05%	-0.13%	1.41%	0.90%
REF	1.32%	0.00%	0.90%	3.31%

	$\tilde{\boldsymbol{\sigma}}$
GE	13.57%
GGB	4.80%
EMB	11.89%
REF	18.19%

First, note how the additional uncertainty in the equilibrium returns of global equities is propagated to the volatility vector $\tilde{\boldsymbol{\sigma}}$ used in the optimization, and conversely for emerging market bonds. Second, the impact on $\tilde{\boldsymbol{\mu}}$ is more complex in this case: The expected return of global equities is increased because the added uncertainty about the equilibrium return pulls the model more toward View 1, which assumes a higher return of global equities. For emerging market bonds, where the expected equilibrium return is estimated with less uncertainty, the return is pulled more towards that equilibrium return than the return implied by View 3.

The resulting allocation is reported in the next table, where a comparison to the results of the more flexible model without views is provided as well.

	$\boldsymbol{\omega}_{FL}$	$\boldsymbol{\omega}_{FL}^{V}$
GE	41.34%	54.89%
GGB	34.50%	35.69%
EMB	13.81%	7.87%
REF	10.35%	1.55%

The overall results are illustrated graphically in Figure 3. The effects of accurately reflecting the uncertainty about expected equilibrium returns in the asset allocation framework are visible, as the more flexible model clearly chooses different weights than the classical Black-Litterman model. Without views, more (un)certainty about expected equilibrium returns tends to increase (decrease) the weight of the respective asset. When views are introduced, effects are more complex, as the uncertainty about expected equilibrium returns is traded off with the uncertainty in the views, and thus an interaction of effects is taken into account. This also shows the advantages of using the more flexible model: These effects are accounted for consistently in a Bayesian framework, presenting the investor with a flexible tool for the asset allocation process.

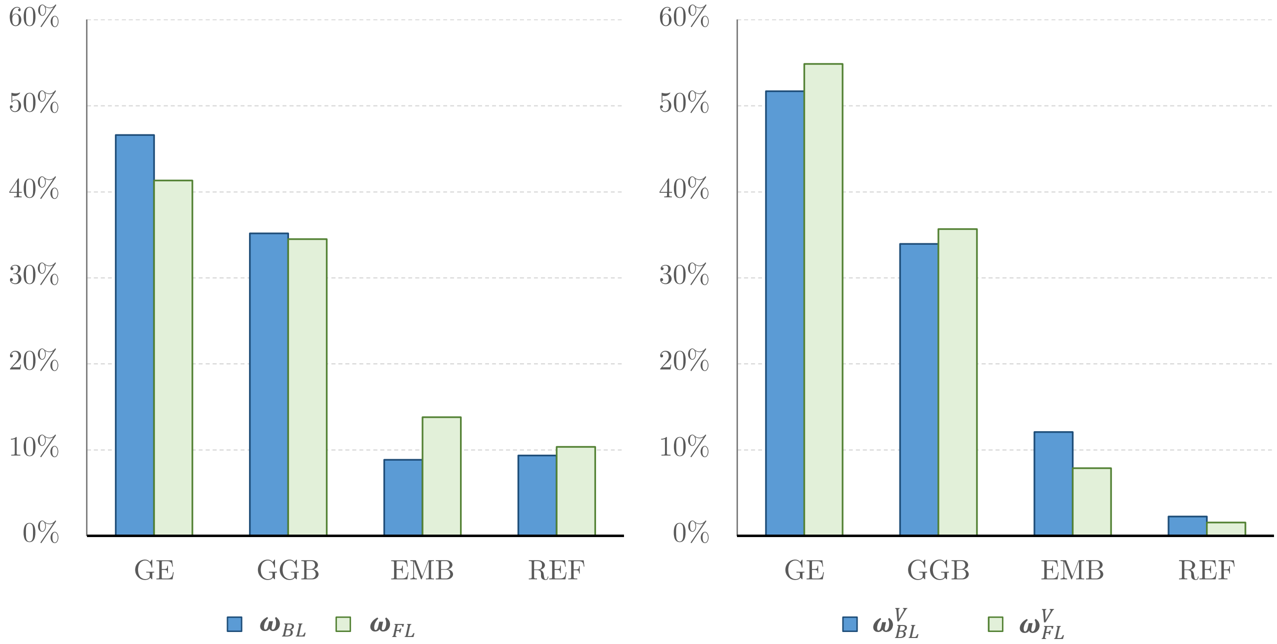

Figure 3: Optimized portfolio weights for the Black-Litterman model and the more flexible model, with and without views.

6 Conclusion

The Black-Litterman model has practical appeal for institutional investors, as it incorporates parameter uncertainty into the asset allocation process. This attenuates one of the major problems of mean-variance optimization. Additionally, the Black-Litterman model allows the investor to specify subjective views that are consistently incorporated into the process, using sophisticated Bayesian methods. The specification of two hyperparameters, $\boldsymbol{\Psi}$ and $\boldsymbol{\Omega}$, controlling the degree of uncertainty about the model-parameters, is however not trivial and introduces rigidity. From a practical point of view, a more flexible way of specifying uncertainty about the expected equilibrium returns would be desirable, as these expectations are derived within a multi-model framework in practice.

We have shown a way to introduce this flexibility into the Black-Litterman model. A new approach is illustrated, where the confidence about expected equilibrium returns can be specified by the investor as well. We also show how the statistical properties of equilibrium models can be used to derive the uncertainty about expected equilibrium returns, and how these can be used as inputs. The results clearly indicate that our proposition has an influence on the resulting asset allocation, and how these effects are propagated through the Bayesian nature of the model.

Further research should firstly focus more on the aspect of incorporating the actual statistical properties of equilibrium models, as various candidate models for different asset classes can be tested and evaluated. This might reduce the subjectivity of setting the hyperparameters in the Black-Litterman model and result in a more rigorous asset allocation process for institutional investors. Secondly, subjective uncertainty, which is not entirely based on empirical or historical distributions could also be incorporated in more sophisticated Bayesian portfolio construction frameworks (e.g. pure non-parametric approaches).

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A Appendix

A.1 Alternative Interpretation of τ

There is an alternative interpretation of τ available when comparing the results of Black-Litterman to the results of *diffuse priors* from Bayesian statistics. The use of *diffuse* or *uninformative* priors is an established methodology to incorporate estimation risk in the allocation process (see, for instance Rachev et al. (2008)). Using Jeffreys' prior (Jeffreys (1961)), it can be shown that the posterior distribution of returns has the following parameters:

$$\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} \qquad \text{and} \qquad \tilde{\boldsymbol{\Sigma}} = \frac{(1+\frac{1}{T})(T-1)}{T-N-2}\boldsymbol{\Sigma} \tag{A.1}$$

To compare this to the Black-Litterman assumption above, consider the Black-Litterman model with no views. In this case,

$$\tilde{\boldsymbol{\mu}} = \boldsymbol{m} = \boldsymbol{\pi} \qquad \text{and} \qquad \tilde{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma} + \boldsymbol{V} = \boldsymbol{\Sigma} + \boldsymbol{\Psi} = (1+\tau)\boldsymbol{\Sigma} \tag{A.2}$$

This allows us to see the connection between using Jeffreys' prior and the Black-Litterman model without subjective views and lends an intuitive interpretation for the parameter τ. In the example above, with $N = 4$ assets, τ becomes a function of T, the number of data points used to estimate the expected returns in the method using Jeffreys' prior:

$$\tau = \frac{(1+\frac{1}{T})(T-1)}{T-6} - 1 \tag{A.3}$$

For instance, setting $T = 36$ (i.e. we estimate expected returns from three years of monthly data), then $\tau = 0.2$. If $T = 120$ (i.e. ten years of data), $\tau = 0.05$.

A.2 Construction of Time Series

For the illustration, we use the following time series (all data obtained from Bloomberg):

	Time Series
GE	MSCI World Total Return (hedged to CHF) minus 3M Libor CHF
GGB	50% FTSE US GBI (constant Duration 6.5 Year) minus 3M US Govt. Bond Yield 50% FTSE Germany GBI (constant Duration 6.5 Year) minus 3M Libor EUR
EMB	Bloomberg Barclays EM USD Aggregate Total Return unhedged minus FTSE US GBI (Duration matched)
REF	FTSE NAREIT Composite Total Return minus 3M US Govt. Bond Yield

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Presserechtliche Verantwortung:
Sonja Wiesel, Hochschulkommunikation und Öffentlichkeitsarbeit
Telefon +49 (9621) 482-3135
Fax +49 (9621) 482-4135
s.wiesel@oth-aw.de

Bestellungen schriftlich erbeten an:
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